Granger-Causal-Priority and Choice of Variables in Vector Autoregressions

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The views expressed here are solely those of the authors and do not necessarily reflect the views of the ECB
The questions that this paper studies

- A researcher wants to forecast $y_i$ or compute impulse responses of $y_i$ with a VAR.

- The researcher has access to data on $y$, where $y_i \subset y$.

- Let $y_J \equiv y \setminus y_i$ and consider $y_j \subseteq y_J$.
  - Does $y_j$ belong in the VAR to be used to forecast $y_i$?
  - Does $y_j$ belong in the VAR to be used to compute impulse responses of $y_i$?
Why study choice of variables in VARs?

- Most applications of VARs involve choice of variables.
  - Most economists have a preference for using the minimal means to get their points across.
  - Most audiences want to understand in simplest terms “where results come from.”

- Typically, choice of variables occurs *informally*.
  - We show that choice of variables can occur *formally* in a straightforward way.
  - Even when choice of variables is informal, it is useful to know what assumptions are implicit and to what extent these assumptions are supported by the data.
How we answer the questions

• We propose a methodology relying on two ingredients:

  1. a restriction on the data generating process followed by \textit{all variables in the dataset}

  2. tools for Bayesian inference.

• We apply the methodology to the case when the variables of interest are output, price level, and short-term interest rate.

  – We obtain remarkably similar findings for the euro area and the United States.
Granger-noncausality and Granger-causal-priority

- Consider \( y(t) = B(L)y(t-1) + u(t) \)

  - \( y_j \) does not Granger-cause \( y_i \) if the coefficients on all lags of \( y_j \) in the equations for \( y_i \) are equal to zero, \( B_{ij}(L) = 0 \)

- \( y_i \) is Granger-causally-prior to \( y_j \) if it is possible to partition all the variables in \( y \) into two subsets, \( y_1 \) and \( y_2 \), such that \( y_i \subseteq y_1 \), \( y_j \subseteq y_2 \), and \( y_2 \) does not Granger-cause \( y_1 \). That is, we have \( B_{12}(L) = 0 \) in

\[
\begin{align*}
  y_i \rightarrow \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} &= \begin{pmatrix} B_{11}(L) & B_{12}(L) \\ B_{21}(L) & B_{22}(L) \end{pmatrix} \begin{pmatrix} y_1(t-1) \\ y_2(t-1) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}
\end{align*}
\]
Granger-causal-priority is stronger than Granger-noncausality

- Consider \( y = \{x, w, z\} \), \( y_i = x \), \( y_j = z \) and suppose we have \( B_{xz} = 0 \) in

\[
\begin{pmatrix}
x(t) \\
w(t) \\
z(t)
\end{pmatrix} = 
\begin{pmatrix}
B_{xx} & B_{wx} & B_{xz} \\
B_{wx} & B_{ww} & B_{wz} \\
B_{xz} & B_{zw} & B_{zz}
\end{pmatrix}
\begin{pmatrix}
x(t-1) \\
w(t-1) \\
z(t-1)
\end{pmatrix} + u(t)
\]

- \( x(t+1) = B_{xx}x(t) + B_{wx}w(t) + u_1(t+1) \), so \( z \) does not Granger-cause \( x \).

- However, \( x(t+2) = \ldots + B_{wx}B_{wz}z(t) + \ldots \)

- \( x \) is Granger-causally-prior to \( z \) if either \( B_{xz} = B_{xw} = 0 \) or \( B_{xz} = B_{wz} = 0 \). All indirect effects from \( z \) to \( x \) are ruled out. The VAR is block recursive.
Granger-causal-priority and choice of variables for forecasting

• If $y_i$ is Granger-causally-prior to $y_j$, $y_j$ does not belong in the VAR to be used to forecast $y_i$.

  – The reason is that the following two VARs give the same forecasts:

  $$y_i \rightarrow \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} B_{11}(L) & 0 \\ B_{21}(L) & B_{22}(L) \end{pmatrix} \begin{pmatrix} y_1(t-1) \\ y_2(t-1) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

  and

  $$y_1(t) = B_{11}(L)y_1(t-1) + u_1(t)$$
Granger-causal-priority and choice of variables for impulse response analysis

- If \( y_i \) is Granger-causally-prior to \( y_j \) and the number of structural shocks that affect \( y_1 \) contemporaneously is equal to the number of variables in \( y_1 \), \( y_j \) does not belong in the VAR to be used to infer impulse responses of \( y_i \).

  - The reason is that the following two structural VARs yield the same impulse responses of \( y_1 \):

\[
\begin{pmatrix}
A_{11}(L) & 0 \\
A_{21}(L) & A_{22}(L)
\end{pmatrix}\begin{pmatrix}
y_1(t) \\
y_2(t)
\end{pmatrix} = \begin{pmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t)
\end{pmatrix}
\]

and

\[
A_{11}(L)y_1(t) = \varepsilon_1(t)
\]
How to evaluate the posterior probability that $y_i$ is GCP to $y_j$?

- We consider a set $\Omega$ of models such that: (i) all models in $\Omega$ are Gaussian VARs with all variables $y$ and at most one Granger-causal-priority restriction, and (ii) $y_i \subseteq y_1$ in each restricted model in $\Omega$.

- Consider $y_j \subseteq y_J$. Let $\Omega^j$ be the subset of $\Omega$ that contains all the models in which $y_j \subseteq y_2$.
  
  - We evaluate the posterior probability of $\Omega^j$, $p\left(\Omega^j|Y, \Omega\right)$.
  
  - Note: If $y_j$ consists of a single variable, $\Omega^j$ and $\Omega \setminus \Omega^j$ have an equal number of elements.
Implementation

• We can evaluate *analytically* the marginal likelihood of any model in \( \Omega \).

  – The reason is that we derive an *analytical* Bayes factor in favor of a Granger-noncausality restriction, \( p(Y|\omega^R)/p(Y|\omega^U) \).

  – We assume: \( p(B, \Sigma|\omega^U) \) is conjugate, and

    \[
p(B(\beta\alpha), \Sigma|\omega^R) = p(B(\beta\alpha), \Sigma|\omega^U, B_{\beta\alpha} = 0) .
    \]

• In our application \( \Omega \) is large. We approximate the posterior probabilities of GCP using the Markov Chain Monte Carlo Model Composition algorithm.
Evaluating the probability of Granger-causal-priority in practice

• In the euro area and in the United States.

• GDP, price level, and short-term interest rate enter $y_i$.

• Thirty-eight macroeconomic and financial variables enter $y_J$: GDP components, government debt, labor market variables, interest rates, monetary aggregates, credit aggregates, exchange rates, commodity prices and other price indexes, housing market variables, stock market variables, survey-based leading indicators.
Prior, sample

• Prior with two components:
  – Sims-Zha prior: Minnesota prior + one-unit-root dummy observation + no-cointegration dummy observations (with hyperparameter values that maximize the marginal likelihood in a training sample).

• Sample: 1999Q1-2012Q4.

• One lag.
Variables associated with low Granger-causal-priority probability

<table>
<thead>
<tr>
<th>Variable</th>
<th>prob.</th>
<th>rank</th>
<th>Variable</th>
<th>prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change of Real Inventories</td>
<td>0.00</td>
<td>1</td>
<td>Oil Price</td>
<td>0.00</td>
</tr>
<tr>
<td>Industrial Confidence</td>
<td>0.00</td>
<td>2</td>
<td>Industrial Confidence</td>
<td>0.00</td>
</tr>
<tr>
<td>2-Year Bond Yield</td>
<td>0.00</td>
<td>3</td>
<td>Change of Real Inventories</td>
<td>0.00</td>
</tr>
<tr>
<td>PMI</td>
<td>0.02</td>
<td>4</td>
<td>Unemployment Rate</td>
<td>0.00</td>
</tr>
<tr>
<td>Lending Rate to NFCs</td>
<td>0.02</td>
<td>5</td>
<td>Bond Spread BBB 7-10 Years</td>
<td>0.00</td>
</tr>
<tr>
<td>Real Investment</td>
<td>0.05</td>
<td>6</td>
<td>2-Year Bond Yield</td>
<td>0.01</td>
</tr>
<tr>
<td>Real Exports</td>
<td>0.05</td>
<td>7</td>
<td>Capacity Utilization</td>
<td>0.03</td>
</tr>
<tr>
<td>Real Imports</td>
<td>0.07</td>
<td>8</td>
<td>Lending Rate to NFCs</td>
<td>0.04</td>
</tr>
<tr>
<td>Mortgage Interest Rate</td>
<td>0.08</td>
<td>9</td>
<td>Hours Worked**</td>
<td>0.08</td>
</tr>
<tr>
<td>Oil Price</td>
<td>0.08</td>
<td>10</td>
<td>Real Investment</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Note: survey-based leading indicators, change in inventories, interest rates on public and private debt, oil price.
## Variables associated with high Granger-causal-priority probability

<table>
<thead>
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<th>prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>M3</td>
<td>0.85</td>
<td>29</td>
<td>Foreign Consumer Prices*</td>
<td>0.74</td>
</tr>
<tr>
<td>M1</td>
<td>0.86</td>
<td>30</td>
<td>Loans to NFCs</td>
<td>0.76</td>
</tr>
<tr>
<td>Stock Market Volatility</td>
<td>0.90</td>
<td>31</td>
<td>Real Exports</td>
<td>0.86</td>
</tr>
<tr>
<td>Loans for House Purchase</td>
<td>0.91</td>
<td>32</td>
<td>Government Debt</td>
<td>0.88</td>
</tr>
<tr>
<td>Stock Market Index</td>
<td>0.92</td>
<td>33</td>
<td>M2</td>
<td>0.90</td>
</tr>
<tr>
<td>Nominal Effective Exchange Rate</td>
<td>0.95</td>
<td>34</td>
<td>Loans for House Purchase</td>
<td>0.95</td>
</tr>
<tr>
<td>Consumer Prices Excl. Energy, Food</td>
<td>0.98</td>
<td>35</td>
<td>Real Housing Investment</td>
<td>0.97</td>
</tr>
<tr>
<td>Government Debt</td>
<td>0.99</td>
<td>36</td>
<td>Consumer Loans</td>
<td>0.99</td>
</tr>
<tr>
<td>House Prices</td>
<td>1.00</td>
<td>37</td>
<td>House Prices</td>
<td>1.00</td>
</tr>
<tr>
<td>Dollar/Euro Exchange Rate</td>
<td>1.00</td>
<td>38</td>
<td>Dollar/Euro Exchange Rate</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Note: house prices, nominal exchange rate, also government debt, broad money and credit aggregates.
Choosing the best VAR

- Evaluating the posterior probabilities yields a unique choice of variables if the probabilities are either zero or one.
  - Asymptotically, the posterior probabilities are either zero or one.

- We assume a zero-one loss function on models, and we search for the VAR with highest marginal likelihood, $\omega^*$. 
<table>
<thead>
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<td>10</td>
<td>Real Investment</td>
<td>0.11</td>
</tr>
<tr>
<td>Real Consumption</td>
<td>0.12</td>
<td>11</td>
<td>10-Year Bond Yield</td>
<td>0.13</td>
</tr>
<tr>
<td>Real Housing Investment</td>
<td>0.18</td>
<td>12</td>
<td>Foreign Short Term Interest Rate*</td>
<td>0.13</td>
</tr>
<tr>
<td>Bond Spread BBB 7-10 Years</td>
<td>0.21</td>
<td>13</td>
<td>PMI</td>
<td>0.14</td>
</tr>
<tr>
<td><strong>Unit Labor Cost</strong></td>
<td>0.23</td>
<td>14</td>
<td>Mortgage Interest Rate</td>
<td>0.16</td>
</tr>
<tr>
<td>Foreign Short Term Interest Rate*</td>
<td>0.26</td>
<td>15</td>
<td>Consumer Confidence</td>
<td>0.17</td>
</tr>
<tr>
<td>Unemployment Rate</td>
<td>0.29</td>
<td>16</td>
<td>Foreign Real GDP*</td>
<td>0.22</td>
</tr>
<tr>
<td>Total Employment</td>
<td>0.40</td>
<td>17</td>
<td>Total Employment</td>
<td>0.28</td>
</tr>
<tr>
<td>Consumer Confidence</td>
<td>0.48</td>
<td>18</td>
<td>PPI</td>
<td>0.30</td>
</tr>
<tr>
<td><strong>Foreign Consumer Prices</strong></td>
<td>0.53</td>
<td>19</td>
<td>Real Imports</td>
<td>0.30</td>
</tr>
<tr>
<td>10-Year Bond Yield</td>
<td>0.54</td>
<td>20</td>
<td>Real Consumption</td>
<td>0.33</td>
</tr>
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Log marginal likelihoods for comparing different models in $\Omega$

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<th>U.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The best model in $\Omega$: $\omega^*$</td>
<td>4825</td>
<td>3516</td>
</tr>
<tr>
<td>The unrestricted model $\omega^U$</td>
<td>4774</td>
<td>3488</td>
</tr>
<tr>
<td>The model “opposite” to $\omega^*$</td>
<td>4740</td>
<td>3380</td>
</tr>
</tbody>
</table>

Lessons:

- The VAR with the best GCP restriction is much better than the unrestricted VAR
- The unrestricted VAR is much better than a VAR with a poorly chosen GCP restriction
Robustness

- Subsamples.

- Prior hyperparameter values.

- Lag length.
Conclusions

- Choice of variables in VARs, for forecasting and for impulse response analysis, connects with the concept of Granger-causal-priority.

- The posterior probability of Granger-causal-priority can be evaluated analytically.

- The following variables belong in the VAR with output, price level, and short-term interest rate:
  
  - survey-based leading indicators, change in inventories, interest rates on public and private debt, oil price.
  
  - The same finding obtains in the euro area and in the United States.
Marginal likelihood vs predictive density score

\[ p(Y|\omega) = p(y_i(1, \ldots, T), y_J(1, \ldots, T)|y_i(0), y_J(0), \omega) = \]
\[ = \prod_{t=0}^{T-1} p(y_i(t + 1), y_J(t + 1)|y_i(0, \ldots, t), y_J(0, \ldots, t), \omega) \]

\[ g(y_i, h = 1|\psi) = \prod_{t=0}^{T-1} p(y_i(t + 1)|y_i(0, \ldots, t), y_J(0, \ldots, t), \omega) \]

- cannot be related to posterior odds on models, wastes information
- \( g(y_i, h = 1|\psi) \) depends on \( h \)
- costly to compute (requires loop over \( t \) )